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The mean electromotive force generated by turbulence in the limit of perfect conductivity

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When homogeneous turbulence acts upon a non-uniform magnetic field $\mathbf{B}_0(\mathbf{x}, t)$, a mean electromotive force $\mathscr{E}(\mathbf{x}, t)$ is in general established owing to the correlation between the velocity field and the fluctuating magnetic field that is generated. The relation between $\mathscr{E}$ and $\mathbf{B}_0$ is linear, and it is generally believed that, when the scale of inhomogeneity of $\mathbf{B}_0$ is sufficiently large, it can be expressed as a series involving successively higher spatial derivatives of $\mathbf{B}_0$:

$$\mathscr{E}_i = \alpha_{il} B_{0l} + \beta_{ilm} \partial B_{0l}/\partial x_m + \ldots,$$

where the tensors α_{il}, β_{ilm}, ... are determined (in principle) by the statistical properties of the turbulence and the magnetic diffusivity λ of the fluid. These tensors are of crucial importance in turbulent dynamo theory. In this paper the question of their asymptotic form in the limit $\lambda \to 0$ is considered. By putting $\lambda = 0$ and assuming that the field is non-random at an initial instant $t = 0$, the developing form of the tensors $\alpha_{il}(t)$ and $\beta_{ilm}(t)$ is determined. The expressions involve time integrals of Lagrangian correlation functions associated with the velocity field, which are comparable in structure with the eddy diffusion tensor in the analogous turbulent diffusion problem (Taylor 1921). Some doubts are expressed concerning the convergence of the time integrals as $t \to \infty$, and it is concluded that a satisfactory treatment of the problem will require the inclusion of weak diffusion effects (as recognized by Parker 1955).

1. Introduction

The object of this paper is to point out certain fundamental difficulties in the theory developed by Parker (1955, 1970, 1971*a–f*) to explain the generation of magnetic fields in astrophysical bodies. The theory is in effect a high conductivity (or small magnetic diffusivity) counterpart of the theory of Krause (1967) and Rädler (1968*a*, *b*).† Certain aspects of the low conductivity limit have been considered by Moffatt (1970*a*).

Let $\mathbf{u}(\mathbf{x}, t)$ be a solenoidal turbulent velocity field, statistically homogeneous in $\mathbf{x}$ and stationary in t, and having zero mean. Let $\mathbf{B}(\mathbf{x}, t)$ be a magnetic field

† These papers, and others on the same topic by F. Krause, K. H. Rädler and M. Steenbeck, have been translated into English by P. H. Roberts & M. Stix, and are available as Technical Report TN/1A-60 (June 1971) of the National Center for Atmospheric Research, Boulder, Colorado.

having no sources other than electric currents within the fluid. The evolution of $\mathbf{B}$ is governed by the solenoidal condition $\nabla . \mathbf{B} = 0$ and the induction equation

$$\partial \mathbf{B}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) + \lambda \nabla^2 \mathbf{B}, \tag{1.1}$$

where λ is the magnetic diffusivity of the fluid. Of particular interest is the evolution of the ensemble average field

$$\mathbf{B}_0(\mathbf{x}, t) = \langle \mathbf{B}(\mathbf{x}, t) \rangle. \tag{1.2}$$

This satisfies the averaged equation

$$\partial \mathbf{B}_0/\partial t = \nabla \wedge \langle \mathbf{u} \wedge \mathbf{b} \rangle + \lambda \nabla^2 \mathbf{B}_0, \tag{1.3}$$

where $\mathbf{b} = \mathbf{B} - \mathbf{B}_0$. A primary aim of turbulent dynamo theory is to obtain an expression for the mean electromotive force $\mathcal{E} = \langle \mathbf{u} \wedge \mathbf{b} \rangle$ in terms of $\mathbf{B}_0$, so that (1.3) can be integrated.

The equation satisfied by $\mathbf{b}$ is

$$\mathscr{L}\{\mathbf{b}\} \equiv \partial \mathbf{b}/\partial t - \nabla \wedge (\mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle) - \lambda \nabla^2 \mathbf{b} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0). \tag{1.4}$$

We shall suppose that

$$\mathbf{b}(\mathbf{x}, 0) = 0 \tag{1.5}$$

so that (1.4) establishes a linear relationship between $\mathbf{b}$ and $\mathbf{B}_0$, and so between $\langle \mathbf{u} \wedge \mathbf{b} \rangle$ and $\mathbf{B}_0$. On the assumption that the scale L of inhomogeneity of B_0 is large ($L \gg l_c$, where l_c is a typical correlation length) this linear relationship may be developed as a series (see, for example, Roberts 1971),

$$\mathcal{E}_i(\mathbf{x}, t) = \langle \mathbf{u} \wedge \mathbf{b} \rangle_i = \alpha_{il}(t) B_{0l} + \beta_{ilm}(t) \, \partial B_{0l}/\partial x_m + \dots, \tag{1.6}$$

where $\alpha_{il}(t)$, $\beta_{ilm}(t)$, ... are pseudo-tensors determined solely by the statistical properties of the turbulence and the parameter λ; their dependence on t arises through the condition (1.5), which clearly implies $\alpha_{il}(0) = 0$, $\beta_{ilm}(0) = 0$.

When the magnetic Reynolds number R_m, defined by

$$R_m = u_0 l_c/\lambda, \quad u_0 = \langle \mathbf{u}^2 \rangle^{\frac{1}{2}}, \tag{1.7}$$

is *small*, it is known that $\alpha_{il}(t), \beta_{ilm}(t), \dots$ quickly settle down to asymptotic steady values; in particular (Moffatt 1970*a*†) in the case of turbulence that is rotationally symmetric,

$$\alpha_{il} = \alpha \delta_{il}, \quad \alpha \sim -\frac{1}{3\lambda} \int k^{-2} F(k) \, dk, \tag{1.8}$$

where $F(k)$ is the helicity spectrum function, with the property

$$\langle \mathbf{u} . \nabla \wedge \mathbf{u} \rangle = \int_0^\infty F(k) \, dk. \tag{1.9}$$

Moreover, in this limit, the term involving β_{ilm} makes a negligible contribution to (1.3) compared with the strong diffusion term $\lambda \nabla^2 \mathbf{B}_0$.

† The factor $\frac{1}{2}$ in equation (3.11) of Moffatt (1970*a*) is incorrect, and should be replaced by $\frac{1}{3}$.

When R_m is large, and in particular in the limit $R_m \to \infty$, the situation is far less clear. First, it is by no means certain that, in the limit $\lambda \to 0$, $\alpha_{il}(t)$ and $\beta_{ilm}(t)$ do settle down to asymptotic steady values as $t \to \infty$. If they do, then on dimensional grounds

$$\alpha \sim u_0, \quad \beta \sim u_0 l_c, \tag{1.10}$$

where α and β are typical components of α_{il}, β_{ilm} – say

$$\alpha = \tfrac{1}{3}\alpha_{ii}, \quad \beta = \tfrac{1}{6}\epsilon_{ijk}\beta_{ijk}; \tag{1.11a, b}$$

the estimates (1.10) arise in models developed by Parker (1955, 1970, 1971*b*). If they do not, then the ultimate values of α_{il}, β_{ilm} must depend on λ through relationships of the form

$$\alpha \sim u_0 f(R_m), \quad \beta \sim u_0 l_c g(R_m) \tag{1.12a, b}$$

and it would then be desirable to obtain the asymptotic form of the functions $f(R_m)$ and $g(R_m)$ for $R_m \to \infty$. In the case of rotationally symmetric turbulence, β is just the eddy diffusivity for magnetic field and a relation of the form (1.12*b*) is certainly not out of the question; a qualitative theory of Moffatt (1961) in fact gave

$$\beta \sim (u_0 l_c) R_m^{\frac{3}{2}} \tag{1.13}$$

in the limiting situation $R_m \to \infty$, $R/R_m \to \infty$, where $R = u_0 l_c/\nu$ and ν is the kinematic viscosity; this theory was based on estimating the spectrum of magnetic fluctuations and calculating the consequent increase in ohmic dissipation of magnetic energy.†

The effectiveness of turbulent dynamos when $R_m \gg 1$ is highly dependent on the appropriate value of the eddy diffusivity, and it is important to be able to distinguish between such alternatives as (1.10*b*) and (1.13). The question of the appropriate asymptotic value of α is likewise of crucial importance in such theories as that of Steenbeck & Krause (1966) for the generation of the solar and terrestrial magnetic fields. The uncertainty in the value of α can be illustrated in physical terms through reference to figure 1, which illustrates the process discussed by Parker (1970). Parker describes as a 'cyclonic event' a velocity field $\mathbf{u}(\mathbf{x}, t)$ which is localized in space and of limited duration and whose linear and angular momenta are related in such a way as to give a definite sense of twisting; it is evident that the total helicity

$$I = \int \mathbf{u}.(\nabla \wedge \mathbf{u})\, d^3\mathbf{x} \tag{1.14}$$

for such a motion must be non-zero (positive in the situation illustrated in figure 1); it can be argued on dynamical grounds that such events are possible, and perhaps probable, in a thermally convecting rotating fluid. The event tends to distort a line of force in the manner indicated in figure 1 (*a*), and a loop appears having a projection on the plane perpendicular to the local (undisturbed) field $\mathbf{B}$.

† Rädler (1968*b*) has made the point that different definitions of eddy diffusivity may well lead to different values.

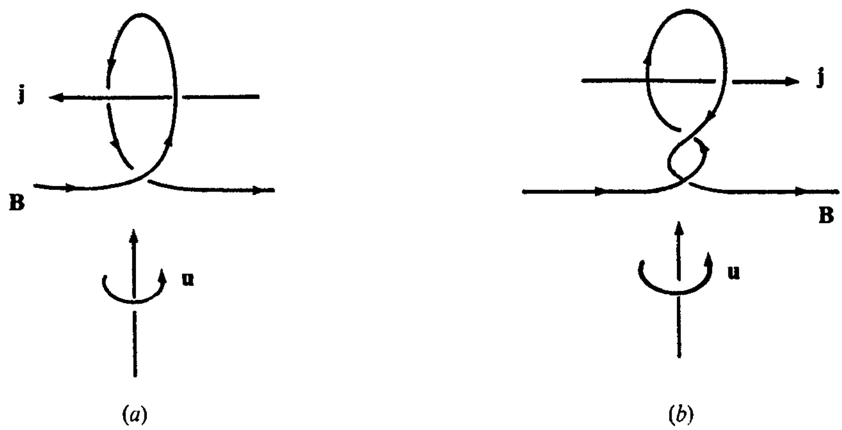

FIGURE 1. Possible distortion of a line of force by a cyclonic event.

From Ampère's law, this loop of field can be conceived as being due to a current $\mathbf{j}$ having a component parallel to $\mathbf{B}$; i.e.

$$\mathbf{j}.\mathbf{B} = \alpha \mathbf{B}^2, \tag{1.15}$$

the α here being identifiable with the α introduced above. If the situation is strongly diffusive or if the event is sufficiently short-lived, then the perturbation field remains weak compared with the unperturbed field, and when $I > 0$, α is evidently negative consistent with the result (1.8).

In the weak diffusion limit however, and for more persistent events, the loop may be twisted any number of times (figure 1*b*) and the appropriate value of α may then apparently be positive or negative. For a random superposition of cyclonic events (with $I > 0$ in each), the asymptotic value of α may depend in a very sensitive way on the precise statistics of the velocity field. If the events are sufficiently short-lived and if different events are uncorrelated, then the limited twist picture of figure 1(*a*) will presumably apply, so that α will be negative; on these specific assumptions, Parker (1970) obtained the following expression for α_{il}:

$$\alpha_{il} = \tfrac{1}{2}\nu_1 \epsilon_{ijk}\left\langle \int X_l(\mathbf{a})\, \partial X_j/\partial a_k\, d^3\mathbf{a} \right\rangle, \tag{1.16}$$

where $\mathbf{X}(\mathbf{a})$ is the total displacement during an event of the particle initially at $\mathbf{a}$, the angular brackets indicate an average over events, and ν_1 represents the mean rate of occurrence of events per unit volume. Discussion of (1.16) is deferred until the comparable formula (3.3) below has been derived.

A possible approach to the weak diffusion limit has been suggested by Roberts (1971); this is simply to restrict attention to the situation $u_0 t_c \ll l_c$ when it may be reasonable to neglect the awkward term

$$\mathbf{G} = \nabla \wedge (\mathbf{u} \wedge \mathbf{b} - \langle \mathbf{u} \wedge \mathbf{b} \rangle) \tag{1.17}$$

in (1.4), and to evaluate $\mathbf{b}$ (in the limit $\lambda \to 0$) from the equation

$$\partial \mathbf{b}/\partial t = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}_0). \tag{1.18}$$

Neglect of the term $\mathbf{G}$ (the 'first-order smoothing approximation') has also been advocated by Lerche (1971*a*, *b*) in an attempt to find solutions of (1.3) without exploiting an expansion of the form (1.6). Unfortunately there appear to be few circumstances when the neglect of $\mathbf{G}$ when $\lambda \to 0$ is justified, unless possibly attention is restricted to a sea of random waves of small amplitude (e.g. inertial waves in a rotating fluid as considered by Moffatt 1970*b*), rather than to turbulence as normally understood. Moreover if $\mathbf{G}$ *is* neglected, then the expression obtained for α in fact vanishes in the limit $\lambda \to 0$. The reason is as follows (cf. the discussion of § 4 of Moffatt 1970*a*): since (1.6) is valid for any field distribution, α_{il} may be calculated on the simple assumption that $\mathbf{B}_0$ is uniform, in which case $\mathscr{E}_i = \alpha_{il} B_{0l}$ is also uniform and so [from (1.1)] $\mathbf{B}_0$ is constant in time. Equation (1.18) then becomes

$$\partial \mathbf{b}/\partial t = \mathbf{B}_0 . \nabla \mathbf{u} \qquad (1.19)$$

and, defining the Fourier transform of $\psi(\mathbf{x}, t)$ by

$$\tilde{\psi}(\mathbf{k}, \omega) = \iint \psi(\mathbf{x}, t) \exp\{i(\mathbf{k} . \mathbf{x} - \omega t)\} \, d^3\mathbf{k} \, d\omega, \qquad (1.20)$$

this becomes

$$\omega \tilde{\mathbf{b}} = -(\mathbf{k} . \mathbf{B}_0) \tilde{\mathbf{u}}. \qquad (1.21)$$

The fact that $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{u}}$ are exactly in phase then implies that

$$\langle \mathbf{u} \wedge \mathbf{b} \rangle = \iint \langle \tilde{\mathbf{u}} \wedge \tilde{\mathbf{b}}^* \rangle \, d^3\mathbf{k} \, d\omega = 0. \qquad (1.22)$$

A little dissipation (i.e. $\lambda \neq 0$), or a little nonlinearity (i.e. $\mathbf{G} \neq 0$) is essential to provide an α-effect. In the case of random inertial waves, this result is evident in the formulae (4.1)–(4.5) of Moffatt (1970*b*).

2. Lagrangian treatment of the induction equation

In the limit $\lambda \to 0$, the solution of (1.1) is best expressed in terms of Lagrangian co-ordinates. Let

$$\mathbf{x} = \mathbf{a} + \mathbf{X}(\mathbf{a}, t) \qquad (2.1)$$

be the equation of the path of the fluid particle satisfying

$$\mathbf{X}(\mathbf{a}, 0) = 0. \qquad (2.2)$$

Let

$$\mathbf{v}(\mathbf{a}, t) = \partial \mathbf{X}/\partial t = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t) \qquad (2.3)$$

be the Lagrangian velocity, and let

$$X_{i,j}(\mathbf{a}, t) = \partial X_i/\partial a_j \qquad (2.4)$$

be the Lagrangian deformation tensor, which gives a measure of the total deformation of an infinitesimal element of fluid initially at the point $\mathbf{a}$ during the time interval $[0, t]$. Also let

$$v_{i,j}(\mathbf{a}, t) = \partial X_{i,j}(\mathbf{a}, t)/\partial t = \partial v_i/\partial a_j \qquad (2.5)$$

be the Lagrangian rate of deformation tensor.

Certain properties of the random function $\mathbf{v}(\mathbf{a}, t)$ in stationary homogeneous turbulence have been obtained by Lumley (1962). In particular, it is known that the single-particle statistics of $\mathbf{v}(\mathbf{a}, t)$ are stationary in time, but that the n-particle statistics are in general non-stationary for $n \leqslant 2$. For example, the correlation coefficient

$$r(\mathbf{a}_1 - \mathbf{a}_2, t) = \langle \mathbf{v}(\mathbf{a}_1, t) . \mathbf{v}(\mathbf{a}_2, t) \rangle / \langle v^2(\mathbf{a}, t) \rangle \tag{2.6}$$

certainly tends to zero as $t \to \infty$, since any two particles ultimately drift infinitely far apart with probability 1. However, $r(\mathbf{a}_1 - \mathbf{a}_2, 0)$ is arbitrarily near to 1 when $|\mathbf{a}_1 - \mathbf{a}_2|$ is sufficiently small (e.g. smaller than the Kolmogorov inner scale). Hence $\partial r / \partial t \neq 0$ in general, and so the two-particle statistics of $\mathbf{v}(\mathbf{a}, t)$ are non-stationary.

This would seem to imply also that, although $\mathbf{v}(\mathbf{a}, t)$ is stationary for fixed $\mathbf{a}$, $v_{i,j}(a, t)$ is in general non-stationary for fixed $\mathbf{a}$ since it involves the joint statistics of $\mathbf{v}$ at two points $\mathbf{a}$ and $\mathbf{a} + \delta\mathbf{a}$ in the limit $\delta\mathbf{a} \to 0$. This conclusion is consistent with the statement

$$\frac{\partial v_i}{\partial a_j} = \frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial a_j} \tag{2.7}$$

and the fact that $\partial u_i / \partial x_k$ is presumably a stationary random function of t even if evaluated on the random path (2.1) while $\partial x_k / \partial a_j$ is certainly a non-stationary random function of t.

The solution of (1.1) with $\lambda = 0$ (cf. Cauchy's solution of the vorticity equation) is

$$B_i(\mathbf{x}, t) = (X_{i,j}(\mathbf{a}, t) + \delta_{ij}) B_j(\mathbf{a}, 0). \tag{2.8}$$

Now

$$\mathscr{E} = \langle \mathbf{u} \wedge \mathbf{b} \rangle = \langle \mathbf{u} \wedge \mathbf{B} \rangle, \tag{2.9}$$

and so, from (2.8),

$$\mathscr{E}_i(\mathbf{x}, t) = \epsilon_{ijk} \langle v_j(\mathbf{a}, t) (X_{k,l}(\mathbf{a}, t) + \delta_{kl}) B_l(\mathbf{a}, 0) \rangle, \tag{2.10}$$

where, on the right-hand side, $\mathbf{a}(\mathbf{x}, t)$ is to be regarded as a random function (for given $\mathbf{x}, t$) varying from one realization of the flow to another. From the initial condition (1.5), (2.10) becomes

$$\mathscr{E}_i(\mathbf{x}, t) = \epsilon_{ijk} \langle v_j(\mathbf{a}, t) (X_{k,l}(\mathbf{a}, t) + \delta_{kl}) B_{0l}(\mathbf{a}, 0) \rangle. \tag{2.11}$$

3. Evaluation of $\alpha_{il}(t)$

In order to evaluate $\alpha_{il}(t)$, as mentioned in § 1, we may simply consider the special case in which

$$\mathbf{B}_0(\mathbf{x}, 0) = \text{constant}, \tag{3.1}$$

in which case $\mathbf{B}_0$ remains also constant in time. Then (2.11) becomes $\mathscr{E}_i = \alpha_{il}(t) B_{0l}$, where

$$\alpha_{il}(t) = \epsilon_{ijk} \langle v_j(\mathbf{a}, t) X_{k,l}(\mathbf{a}, t) \rangle \tag{3.2}$$

since $\langle \mathbf{v} \rangle = 0$. It is instructive to express this as a time integral,

$$\alpha_{il}(t) = \epsilon_{ijk} \int_0^t \langle v_j(\mathbf{a}, t) v_{k,l}(\mathbf{a}, \tau) \rangle \, d\tau \tag{3.3}$$

and to compare with Taylor's (1921) expression for the eddy diffusivity tensor for a passive scalar field

$$D_{il}(t) = \int_0^t \langle v_i(\mathbf{a},t)\, v_l(\mathbf{a},\tau)\rangle\, d\tau. \tag{3.4}$$

Owing to the stationarity of $\mathbf{v}(\mathbf{a},t)$ for fixed $\mathbf{a}$, the Lagrangian correlation

$$R_{il}(t-\tau) = \langle v_i(\mathbf{a},t)\, v_l(\mathbf{a},\tau)\rangle \tag{3.5}$$

is a function only of the time difference $t-\tau$, and provided only that $R_{il}(t_1)$ is integrable in $[0,\infty]$,

$$D_{il} \sim \int_0^\infty R_{il}(t_1)\, dt_1 \quad \text{as} \quad t \to \infty. \tag{3.6}$$

The Lagrangian correlation time t_c may be defined by

$$R_{ii}(0)\, t_c = \int_0^\infty R_{ii}(t)\, dt \tag{3.7}$$

and the asymptotic result (3.6) is accurate for $t \gg t_c$.

It is tempting to apply the same formalism to (3.3), but we are now faced with the difficulty that $v_{k,l}(\mathbf{a},\tau)$ is not stationary in τ (§ 2), so that the Lagrangian correlation

$$S_{jkl}(t,\tau) = \langle v_j(\mathbf{a},t)\, v_{k,l}(\mathbf{a},\tau)\rangle \tag{3.8}$$

may depend on t and τ independently (and not just on the difference $t-\tau$). It is reasonable to suppose that

$$S_{jkl}(t,\tau) \to 0 \quad \text{at} \quad t-\tau \to \infty \tag{3.9}$$

but this is not sufficient to guarantee the convergence of the integral (3.3) as $t \to \infty$. For example, if

$$S_{jkl}(t,\tau) = C_{jkl} \cos \omega t\, e^{-k(t-\tau)}, \tag{3.10}$$

(which is not implausible bearing in mind the discussion in § 1 of the physical process represented by figure 1), then

$$\alpha_{il}(t) = \epsilon_{ijk} C_{jkl} k^{-1} \cos \omega t, \tag{3.11}$$

which does not settle down to a constant value as $t \to \infty$.

Unfortunately, the result (3.3) is of interest only if the integral *does* converge as $t \to \infty$; for if not, then in a situation in which $\mathbf{B}_0(\mathbf{x},t)$ is changing with time, the e.m.f. $\mathscr{E}(\mathbf{x},t)$ would depend on values of $\mathbf{B}_0(\mathbf{x},t')$ for arbitrarily large values of $t-t'$, and the assumptions underlying an instantaneous expansion of the form (1.6) would be invalid. Of course it is possible to conceive of kinematically possible velocity fields (as in effect done by Parker 1955) which get round the difficulty of the possible divergence of (3.3); for example, suppose that $\mathbf{v}(\mathbf{a},t)$ is first 'switched on' for a time of order t_c, then switched off for a long time of order $t_\lambda = O(l_0^2/\lambda)$ to allow the small-scale field to be eliminated by diffusion, and that the whole process is then repeated, the statistical properties of $\mathbf{v}(\mathbf{a},t)$ being periodic with the period of the cycle. Then there is an effective cut-off when $t = O(t_c)$ to the integral (3.3). The need to invoke molecular diffusion was emphasized by Parker (1955) and by Parker & Krook (1956), but the more formal treatment of Parker (1970) in fact makes no appeal to effects of molecular diffusivity and the expression (1.16) obtained by Parker (1970) in consequence shows no explicit dependence on λ.

The expression (3.3) may be compared with Parker's expression (1.16), with which it clearly has a certain formal resemblance. The advantage of (3.3) is that this expression is *determinate for any turbulent velocity field* and is (in principle) measurable by following the continuous deformation of small dye spots in a turbulent flow and averaging over spots.† The expression (1.16) is defined only in terms of a rather artificial model of turbulence (random cyclonic events) and it is not measurable, even in principle; the concept of a mean rate of occurrence of events is particularly elusive for a fully developed turbulent flow in which events merge in a distinctly nonlinear manner in both space and time. Moreover, the expression (1.15) conceals the possibility that $\alpha_{il}(t)$ as calculated on a $\lambda = 0$ basis may in fact diverge as $t \to \infty$, and that finite λ effects may have to be considered in this eventuality to determine the correct asymptotic behaviour for large t.

The structure of (3.3) is of particular interest when the turbulence is rotationally symmetric, in which case $\alpha_{il}(t) = \alpha(t)\,\delta_{il}$, where

$$\alpha(t) = \tfrac{1}{3}\alpha_{ii}(t) = -\frac{1}{3}\int_0^t \langle \mathbf{v}(\mathbf{a},t).\nabla_{\mathbf{a}} \wedge \mathbf{v}(\mathbf{a},\tau)\rangle\, d\tau. \qquad (3.12)$$

Note the appearance of the 'Lagrangian helicity correlation' under the integral sign. If $\langle \mathbf{u}.\nabla \wedge \mathbf{u}\rangle > 0$ say, then it is certain that $\alpha(t) < 0$ for $t \ll t_c$; but there seems no obvious reason why $\alpha(t)$ as defined by (3.12) should remain negative as t increases. The expression (3.12) may be compared with the expression (1.8) obtained in the strong diffusion limit.

4. Evaluation of $\beta_{ilm}(t)$

From (1.3) and (1.6),

$$\partial B_{0i}/\partial t = \epsilon_{ijk}\alpha_{kl}B_{0l,j} + \epsilon_{ijk}\beta_{klm}B_{0l,jm} + \ldots + \lambda B_{0i,kk}, \qquad (4.1)$$

and so
$$\partial B_{0i,p}/\partial t = \epsilon_{ijk}\alpha_{kl}B_{0l,jp} + \epsilon_{ijk}\beta_{klm}B_{0l,jmp} + \ldots + \lambda\nabla^2 B_{0l,pkk}. \qquad (4.2)$$

Hence if the field gradient tensor $B_{0i,p}$ is uniform initially then, according to (4.2), in strictly homogeneous turbulence $B_{0i,p}$ remains constant in time. It then follows from (4.1) that at any point $\mathbf{x}$

$$B_{0i}(\mathbf{x},t) = B_{0i}(\mathbf{x},0) + \epsilon_{ijk}B_{0l,j}\int_0^t \alpha_{kl}(\tau)\,d\tau. \qquad (4.3)$$

In order to evaluate $\beta_{ilm}(t)$, we may suppose that $B_{0l,m}$ is uniform (and so constant), and that

$$B_{0l}(\mathbf{a},0) = B_{0l}(\mathbf{x},0) + (\mathbf{x}-\mathbf{a})_m B_{0l,m}. \qquad (4.4)$$

† A numerical experiment to test the large time behaviour of components of $S_{jkl}(t,\tau)$ or of the function $\alpha(t)$ defined by (3.12) would be of great interest.

Substitution in (2.11) then gives

$$\mathscr{E}_i(\mathbf{x},t) = \alpha_{il}(t)\,B_{0l}(\mathbf{x},0) + \hat{\beta}_{ilm}(t)\,B_{0l,m}, \tag{4.5}$$

where $\alpha_{il}(t)$ is defined by (3.3), $\hat{\beta}_{ilm}(t)$ is given by

$$\begin{aligned}\hat{\beta}_{ilm}(t) &= \epsilon_{ijk}\langle v_j(\mathbf{a},t)\,(X_{k,l}(\mathbf{a},t)+\delta_{kl})\,X_m(\mathbf{a},t)\rangle \\ &= \epsilon_{ijl}\,D_{jm}(t) + \epsilon_{ijk}\int_0^t\int_0^t \langle v_j(\mathbf{a},t)\,v_{k,l}(\mathbf{a},\tau_1)\,v_m(\mathbf{a},\tau_2)\rangle\,d\tau_1\,d\tau_2\end{aligned} \tag{4.6}$$

and $D_{jm}(t)$ is given by (3.4) above. Using (4.3), (4.5) may be expressed in the required form

$$\mathscr{E}_i(\mathbf{x},t) = \alpha_{il}(t)\,B_{0l}(\mathbf{x},t) + \beta_{ilm}(t)\,B_{0l,m}, \tag{4.7}$$

where

$$\beta_{ilm}(t) = \hat{\beta}_{ilm}(t) - \epsilon_{pmk}\int_0^t \alpha_{kl}(\tau)\,\alpha_{ip}(t)\,dt. \tag{4.8}$$

This expression for $\beta_{ilm}(t)$ now reveals a double difficulty. First, the double integral of the triple Lagrangian correlation in (4.6) is even less likely to converge than the integral (3.3) defining $\alpha_{il}(t)$. Second, if $\alpha_{kl}(t)$ tends to a non-zero constant tensor as $t \to \infty$ (as envisaged by Parker 1970) then the integral in (4.8) *certainly* diverges, so that the ultimate level of β_{ilm} can *only* be determined by considering the effects of weak diffusion.

It is hard to escape the conclusion that when $\lambda \to 0$ the eddy diffusivity β defined by (1.11*b*) in general increases without limit, and that a simple asymptotic relation of the form (1.10*b*) is unlikely to be correct. This conclusion is at variance with that of Parker (1971*b*) who argued that the eddy diffusivity for a magnetic field in the limit $\lambda \to 0$ should be identical with that for a scalar field (such as temperature) viz. $\frac{1}{3}D_{ii}(\infty)$. Parker's argument, however, relied implicitly on the splitting of the correlation

$$\langle v_j(\mathbf{a},t)\,X_m(\mathbf{a},t)\,X_{m,l}(\mathbf{a},t)\rangle \to \langle v_j(\mathbf{a},t)\,X_m(\mathbf{a},t)\rangle\,\langle X_{m,l}(\mathbf{a},t)\rangle, \tag{4.9}$$

a step for which there is no apparent justification. (If this step were justifiable, then it is hard to see why the simpler correlation splitting

$$\langle v_j(\mathbf{a},t)\,X_{m,l}(\mathbf{a},t)\rangle \to \langle v_j(\mathbf{a},t)\rangle\,\langle X_{m,l}(\mathbf{a},t)\rangle$$

would not be equally justifiable, in which case the tensor $\alpha_{il}(t)$ would necessarily vanish for all t.)

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